II. EVALUATION OF RESPONSE

A. GEOMETRY OF A SUBMARINE

Froude (1877) [Ref. 4] introduced the concept of a ship with a forward end called the “entrance”, a parallel middle body, and an after end called the “run”. Chapman (1768) [Ref. 4] introduced the concept of a ship hull with the entrance a portion of an ellipsoid of revolution, and with the run a portion of a parabola of revolution. This concept is tailor-made for use in calculating volumes of modern submarine hulls, as described by Jackson (1983) [Ref. 4]. It was developed by assuming a body of revolution with a length/diameter (L/D) ratio of six and a maximum diameter at $0.4L$. The entrance has a length , L_f , of 2.4 diameters. The run or after end has a length, L_a , of 3.6 diameters. The entrance can be calculated as an ellipsoid of revolution, and the run as a paraboloid of revolution which is rotated about a line parallel to the center-line. The equations of the offsets for each are given below. The hull radius at each station can be found by multiplying the offsets by half the maximum diameter, $D/2$.

If one were to use equations for true ellipsoids and parabolas, the entrance and the run would be too fine for a modern submarine. The displacement can be increased by using larger exponents (n_f and n_a), as in Equations (1) and (2). If even more displacement is required, a parallel middle body of cylindrical shape can be inserted at the maximum diameter. The prismatic coefficient, C_p, is used to calculate volumes and for a cylinder the prismatic coefficient is 1. For a submarine-like body the prismatic coefficient can be evaluated in terms of its geometry. Using the above concept, the length of the parallel middle body (PMB) is the length overall (LOA) less $6D$, that is, LOA-$6D$.

$$y_f = \frac{D}{2}\left[1-\left(\frac{x_f}{L_f}\right)^{n_f}\right]^{1/n_f} \quad \textbf{(1)}$$

$$y_a = \frac{D}{2}\left[1-\left(\frac{x_a}{L_a}\right)^{n_a}\right] \quad \textbf{(2)}$$

Here x_f and x_a are the distances from the maximum diameter. With these concepts, a very simple method of calculating the volume of the entire hull can be developed. This is true for the ends separately and for the PMB. Let V_f, V_a, and V_{PMB} denote, respectively, volume of the entrance, the run, and the parallel middle body, and let C_f and C_a be the prismatic coefficients. The resulting equations are

$$V_f = \frac{\pi D^2}{4}\left(C_{pf}\, 2.4D\right) \quad \textbf{(3)}$$

$$V_a = \frac{\pi D^2}{4}\left(C_{pa}\, 3.6D\right) \quad \textbf{(4)}$$

$$V_{PMB} = \frac{\pi D^2}{4}\left(L-6D\right) \quad \textbf{(5)}$$

The above can be combined into the following

$$V = \frac{\pi D^3}{4}\left[3.6C_{pa} + \frac{L}{D} - 6 + 2.4C_{pf}\right] \quad \textbf{(6)}$$

Figure 1 illustrates this concept [Ref. 4]. In this study our model submarine's LOA is 109.75 m. (360 ft.), D is 9.15 m. (30 ft.), and the exponents n_a and n_f are 3.0.

B. REGULAR WAVE RESPONSE

In this study our main concern is the effects of surface waves on a near-surface submarine vertical motions. The assumptions here are that the fluid is ideal and the wave

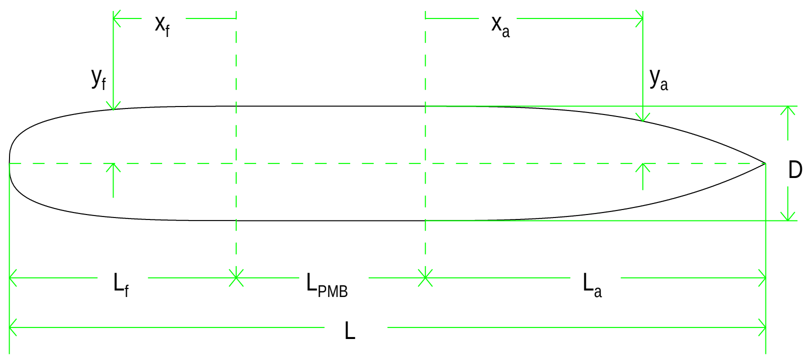

Figure 1. Submarine Geometry [Ref. 4]

and body motions are sufficiently small to linearize. Sea waves and ship motions are based on potential flow theory. In the simplest case it may be assumed that the waves incident upon the body are plane progressive waves of small amplitude, with sinusoidal time dependence. The solution to the water velocity distribution associated with the wave can be simplified to deep and shallow water approximations. When the wave length is greater than 20 times the depth, then shallow water approximations apply and depth becomes the controlling factor. The horizontal component of the velocity is not a function of depth, but is constant from top to bottom. The vertical component of water motion decays linearly from its maximum at the surface to zero at the bottom. The pressure under a shallow-water wave also is not a function of depth, but is just the hydrostatic pressure due to the amount of water above. When the water depth is greater than one quarter the wavelength then the deep water approximations apply and the water depth becomes unimportant. The horizontal and vertical components of velocity are equal and the orbits become circles which decrease exponentially as a function of depth. Their motions become negligible at a depth equal to one half the wavelength [Ref. 5]. Since we are interested in the vertical motion we consider the motions of a body that is allowed to heave and pitch only. Such motions are usually decoupled, for typical ships, from the horizontal plane motions in sway or yaw. The final coupled form of the heave and pitch equations of a ship in regular wave is then

$$(m + A_{33})\ddot{\eta}_3 + B_{35}\dot{\eta}_3 + C_{33}\eta_3 + A_{35}\ddot{\eta}_5 + B_{35}\dot{\eta}_5 + C_{35}\eta_5 = F_3 e^{i\omega t} \quad \textbf{(7)}$$

$$(I_{55} + A_{55})\ddot{\eta}_5 + B_{55}\dot{\eta}_5 + C_{55}\eta_5 + A_{53}\ddot{\eta}_3 + B_{53}\dot{\eta}_3 + C_{53}\eta_3 = F_5 e^{i\omega t} \quad \textbf{(8)}$$

where m is the ship's mass and I_{55} the mass moment of inertia with respect to the y axis. The A_{jk} terms correspond to added mass. The B_{jk} terms correspond to hydrodynamic damping. Terms involving the coefficients ; i.e., C_{33} , C_{55} , and C_{55} are related to tons per cm immersion, change in displacement per cm, and moment to trim one cm, respectively. The right hand side represents the heave, F_3 , Froude-Krylov and diffraction excitation forces. F_3 and F_5 are taken to be the complex exciting force and moment amplitudes, containing both amplitude and phase information. The previous equations of motion are valid for a ship with zero forward speed. If the ship possesses a forward speed U , this can be assumed, within linearity, constant. The only change in such a case is in the frequency ω due to a Doppler shift effect. In linear theory, the harmonic responses of the vessel, $\eta_j(t)$, will be proportional to the amplitude of the exciting forces and at the same frequency, which is now ω_e (wave frequency of encounter) instead of ω. Consequently ship motions will have the form

$$\eta_j(t) = \overline{\eta}_j e^{i\omega_e t},$$

$$\dot{\eta}_j(t) = i\omega_e \overline{\eta}_j e^{i\omega_e t},\ j = 3{,}5 \qquad \textbf{(9)}$$

$$\ddot{\eta}_j(t) = -\omega_e^2 \overline{\eta}_j e^{i\omega_e t},$$

where $\overline{\eta}_j$ is the complex response amplitude, and *j*=3 for heave and *j*=5 for pitch. Substituting equations (9) in (7) and (8), the $e^{i\omega_e t}$ terms cancel out and the resulting equations are

$$\left[-\omega_e^2\left(m + A_{33}\right) + i\omega_e B_{33} + C_{33}\right]\overline{\eta}_3 + \left(-\omega_e^2 A_{35} + i\omega_e B_{35} + C_{35}\right)\overline{\eta}_5 = F_3, \qquad \textbf{(10)}$$

$$\left[-\omega_e^2\left(I_{55} + A_{55}\right) + i\omega_e B_{55} + C_{55}\right]\overline{\eta}_5 + \left(-\omega_e^2 A_{53} + i\omega_e B_{53} + C_{53}\right)\overline{\eta}_3 = F_5, \qquad \textbf{(11)}$$

In equations (10) and (11) the origin is at the center of gravity, which is assumed to lie on the waterline. In the more general case, the term $m\ddot{\eta}_3$ is substituted by $m\ddot{\eta}_3 - mx_G\ddot{\eta}_5$, and the term $I_{55}\ddot{\eta}_5$ by $I_{55}\ddot{\eta}_5 + m(z_G\ddot{\eta}_1 - x_G\ddot{\eta}_3)$, where η_1 is the surge motion amplitude. In other words, a coordinate coupling is introduced.

The determination of the coefficients and exciting forces and moments amplitudes represents the major problem in ship motions calculations. The problem can be simplified by applying a strip theory approach, where the ship is divided into transverse strips, or segments. The added mass and damping for each strip are relatively easily calculated, using two dimensional potential theory or by suitable two dimensional experiments. The sectional values are appropriately combined to yield values for A_{jk}, B_{jk}, C_{jk}, and F_j.

To solve equations (10) and (11) for the complex amplitudes, the equations are written in the form

$$P\overline{\eta}_3 + Q\overline{\eta}_5 = F_3, \quad \textbf{(12)}$$

$$R\overline{\eta}_3 + S\overline{\eta}_5 = F_5, \quad \textbf{(13)}$$

where

$$P = -\omega_e^2\left(m + A_{33}\right) + i\omega_e B_{33} + C_{33,}$$

$$Q = -\omega_e^2 A_{35} + i\omega_e B_{35} + C_{35,}$$

$$R = -\omega_e^2 A_{53} + i\omega_e B_{53} + C_{53,}$$

$$S = -\omega_e^2\left(I_{55} + A_{55}\right) + i\omega_e B_{55} + C_{55,}$$

The solutions to the coupled equations (12) and (13) is then given by

$$\overline{\eta}_3 = \frac{F_3 S - F_5 Q}{PS - QR} \ , \quad \textbf{(14)}$$

$$\overline{\eta}_5 = \frac{F_5 P - F_3 R}{PS - QR} \ . \quad \textbf{(15)}$$

The ratio (η_j/A) is a quantity of fundamental significance, where A is wave amplitude, and is defined by $Z_j(\omega,U,\theta)$. Physically, this is the complex amplitude of body motion in the j-th mode, in response to an incident wave of unit amplitude, frequency ω, and direction θ. The body itself moves with forward speed U. This ratio is generally known as the *transfer function,* or the *response amplitude operator,* RAO. The RAO can be calculated once the added mass, damping, exciting, and hydrostatic forces are known.

The absolute vertical displacement at a point x along the length of the hull, due to heave and pitch is given by

$$\xi_{VA} = \eta_3 - x\eta_5 \ , \quad \textbf{(16)}$$

and since η_3 and η_5 are the complex amplitudes in heave and pitch motion, respectively, ξ_{VA} contains both magnitude and phase information. Particularly of interest in this study is the relative vertical motion between a point in the ship and the surface of the encountered wave. The relative motion in regular waves is found by subtracting the free surface motion from the vertical ship motion at the desired point, taking account of their phase relationship. The free surface motion is composed of the incident wave, the diffracted wave, the radiated wave, and the Kelvin wave due to the ship's steady forward speed. The traditional assumption is that the principal component is the incident wave; i.e., the incident wave is not distorted by the presence of the ship. Then the amplitude of the relative vertical motion in general is given by

$$\xi_{VR} = \xi_{VA} - Ae^{ikx} \ , \qquad \textbf{(17)}$$

where *A* is the wave amplitude and *k* is the wave number. Then the RAO which requires only the scalar or absolute magnitude is

$$\left|\frac{\xi_{VR}}{A}\right| = \left|\frac{\eta_3 - x\eta_5}{A} - e^{ikx}\right|. \qquad \textbf{(18)}$$

The significance of the relative motion response is that the moments of their spectrum provide probability measures related to anticipated deck wetness, bow slamming or particularly for our study sail broaching and periscope submergence.

Figure 2 and 3 show RAO amplitudes and phases versus wave to ship length ratio of our model submarine's heave, pitch, and relative vertical motion at 3 submarine diameter depth and 5 knots forward speed when *A* equals 5 feet.

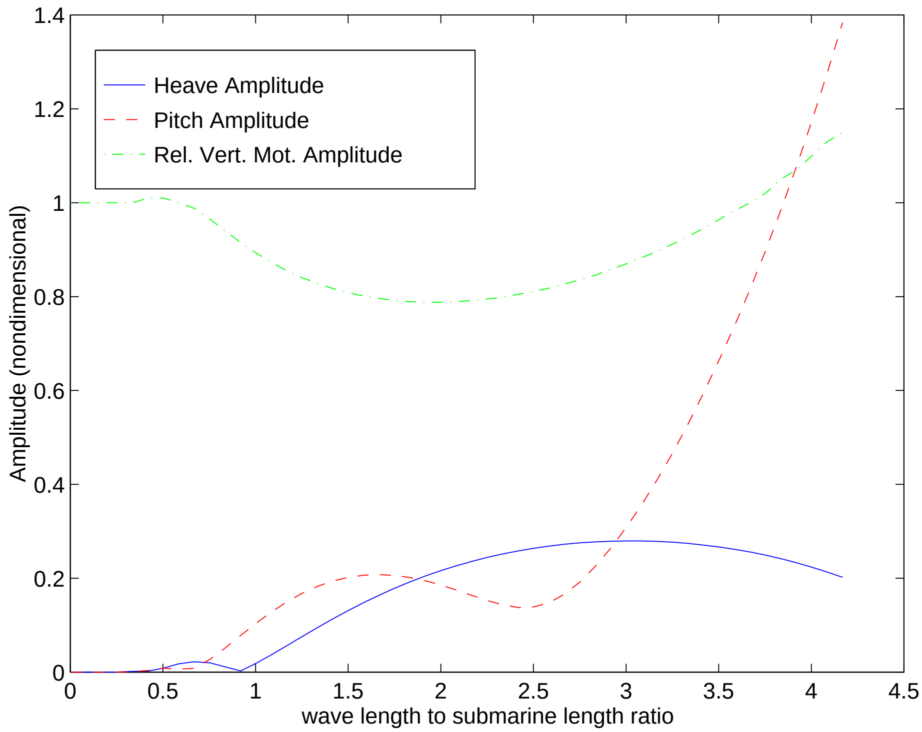

Figure 2. Amplitude of RAO for heave/pitch and relative motion

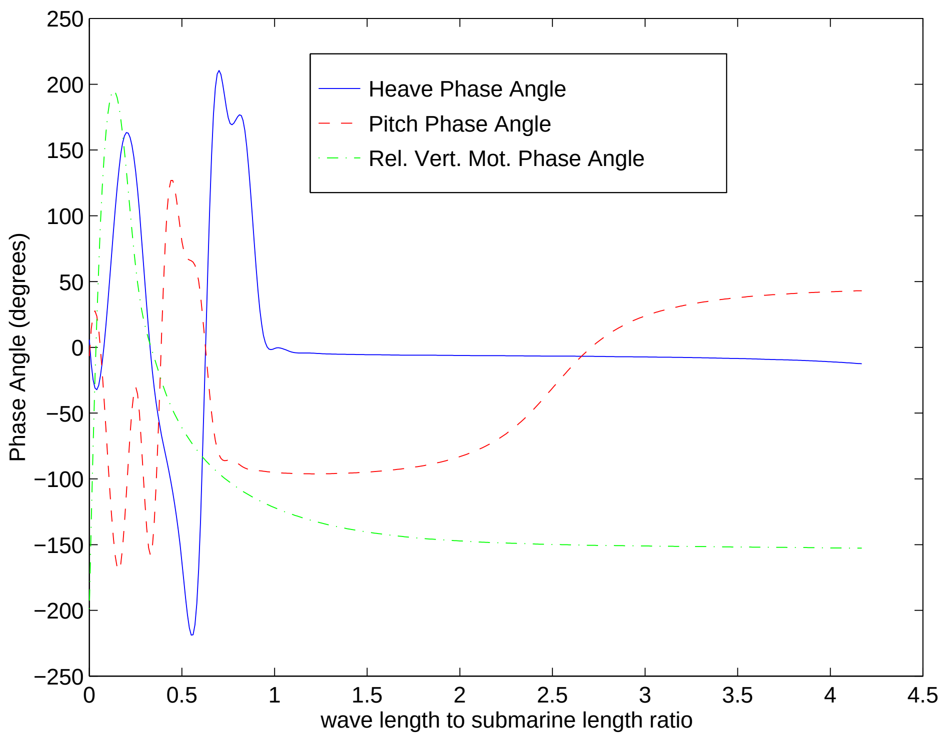

Figure 3. Phase Angle of RAO for heave/pitch and relative motion

C. IRREGULAR WAVE RESPONSE

Wave patterns in an open sea are ever changing with time and space, in a manner that appears to defy analysis be it linear or second order Stokes. Ambient waves on the surface of the sea are dispersive as well as random. Random refers to the character of the wave height distribution. The continuous distribution of sinusoidal waves have continuously distributed amplitude and phase so that in summation the variation of wave height with time is not systematic in any respect, but random. The practically useful data extractable from a random wave record $h(t)$ is its spectral density, $S(\omega)$. The random $h(t)$ record is processed in such a way to produce a curve of $S(\omega)$ versus wave frequency, ω. The spectral density is obtained from a wave height record taken over a time period for which the sea conditions are assumed to be unchanging, in an average sense (stationary). This corresponds to a certain sea state. The function $S(\omega,\theta)$ is called the spectral energy density or simply the energy spectrum. More specifically, this is a directional energy spectrum; it can be integrated over all wave directions to give the frequency spectrum

$$S(\omega) = \int_0^{2\pi} S(\omega,\theta) d\theta \ . \qquad \textbf{(19)}$$

Usually in the fields of ocean engineering and naval architecture it is customary to assume that the waves are long crested which means the fluid motion is two dimensional and the wave crests are parallel. With such a simplification it is possible to use existing information for the frequency spectrum (19), which is based on a combination of theory and full scale observations.

For most purposes we are interested primarily in the larger waves. The most common parameter that takes this into account is the significant wave height, $H_{1/3}$, defined as the average of the highest one third of all waves. This can be computed from

$$H_{1/3} = 4.0(m_0)^{1/2} . \qquad \textbf{(20)}$$

In this equation, m_0 is the area under the spectrum $S(\omega)$ integrated over the entire range of frequencies ω. An average frequency of the spectrum can be defined as the expected number of zero upcrossings per unit time, that is, the number of times the wave amplitude passes through zero with positive slope. The final result here is

$$\omega_z = \left(\frac{m_2}{m_0} \right)^{1/2} . \qquad \textbf{(21)}$$

The average period between zero upcrossings is

$$T_z = \frac{2\pi}{\omega_z} = 2\pi \sqrt{\frac{m_0}{m_2}} \qquad \textbf{(22)}$$

More meaningful frequency parameters can be obtained from the set of moments, which depend on spectrum shape

$$m_n = \int_0^\infty \omega^n S(\omega) d\omega \ , \qquad n=0,1,2,... \qquad \textbf{(23)}$$

In particular, the area, m_0, is the variance or the total energy of the spectrum. Also m_2 is variance of velocity and m_4 is variance of acceleration.

A good model for fully developed seas is the classical Pierson-Moskowitz spectrum. This spectral form depends upon a single parameter which is the significant wave height. It is intended to represent point spectrum of a fully-developed sea. Fetch and duration are assumed to sufficiently large so that the sea has reached steady state, in a statistical sense. This spectral family should be recognized as an asymptotic form, reached after an extended period of steady wind, with no contamination from an underlying swell. Using the spectral family, along with the similarity theory of S. A. Kitaigorodskii, Pierson and Moskowitz (1964) [Ref. 6] arrived at the following analytical formulation for ideal sea spectra,

$$S_1^+(\omega) = \frac{0.0081 g^2}{\omega^5} \exp\left[-0.032 \left(\frac{g}{H_{1/3}\omega^2} \right)^2 \right] , \qquad \textbf{(24)}$$

where

$S_1^+(\omega)$ = one-sided incident wave spectrum

g = acceleration of gravity

$H_{1/3}$ = significant wave height

ω = wave frequency

In Figure 4 we can observe typical Pierson-Moskowitz wave spectra for 5 m. significant wave height.

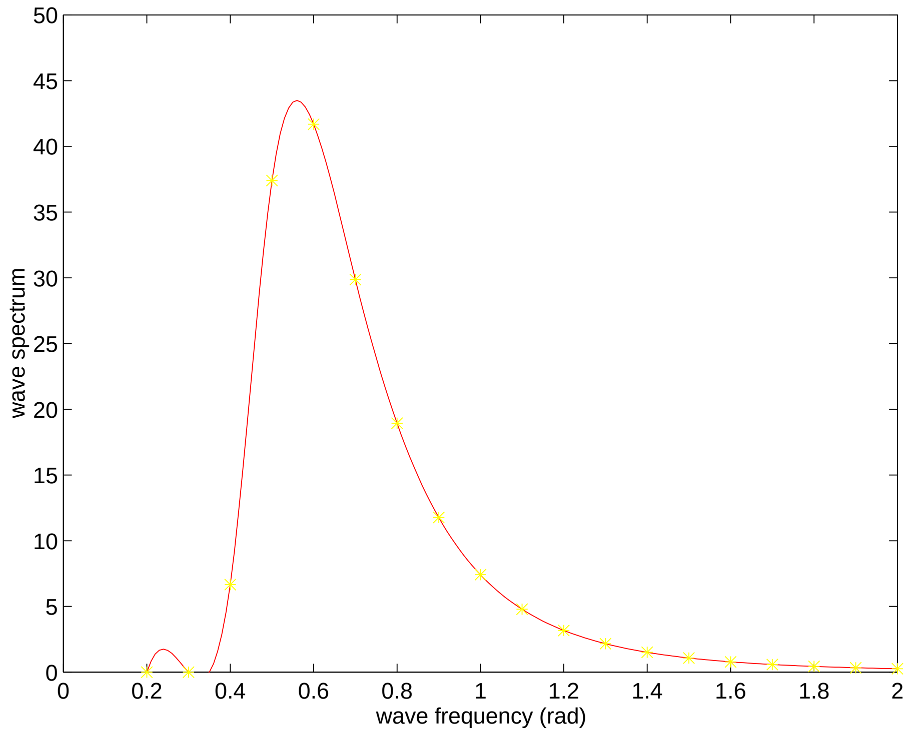

Figure 4. Typical Pierson-Moskowitz wave spectra

Any conclusions drawn on the seakeeping behavior of a ship based on the critical examination of motion response in regular waves can, at best, assume only academic significance. The establishment of the seakeeping behavior of a ship has to be done in a realistic seaway. With the spectral description of sea waves given before, we can return to the subject of body motions and generalize the results of regular harmonic waves. If the sea waves are described by the random distribution , and if the response of the body to each component wave is defined by a response amplitude operator $Z(\omega,\theta)$, the body response will be

$$\eta_j(t) = \Re \iint Z_j(\omega,\theta) e^{i\omega t} dA(\omega,\theta) \ . \qquad \textbf{(25)}$$

The principal assumption here is that linear superposition applies, as it must in any event for the underlying development of the RAO and the spectrum.

Like the waves themselves, the response (25) is a random variable. The statistic of the body response are identical to the wave statistics, except that the wave energy spectrum S is multiplied by the square of the RAO (this is a property of linear systems). Thus, if the subscript R represents any body response, we have

$$S_R(\omega) = \left|Z_R(\omega)\right|^2 S(\omega) \ , \qquad \textbf{(26)}$$

where $Z_R(\omega)$ is the RAO of the response R, and $S(\omega)$ the spectrum of the seaway. Equation (26) can then be utilized to obtain the spectrum of the response R. Figure 5 displays the spectrum of response of the relative vertical motion at the top of our model submarine's sail while submarine's forward speed is 5 Knots and it is at 3 submarine diameter depth. Also seaway is modeled by Pierson-Moskowitz spectrum with 5 m. significant wave height and head seas.

To a large extent, equation (26) provides the justification for studying regular wave responses. The transfer function $Z_R(\omega)$ is valid not only in regular waves, where it has been derived, but also in a superposition of regular waves, and ultimately in a spectrum of random waves. Generally speaking, a vessel with favorable response characteristics in regular waves will be good in irregular waves, and vice versa.

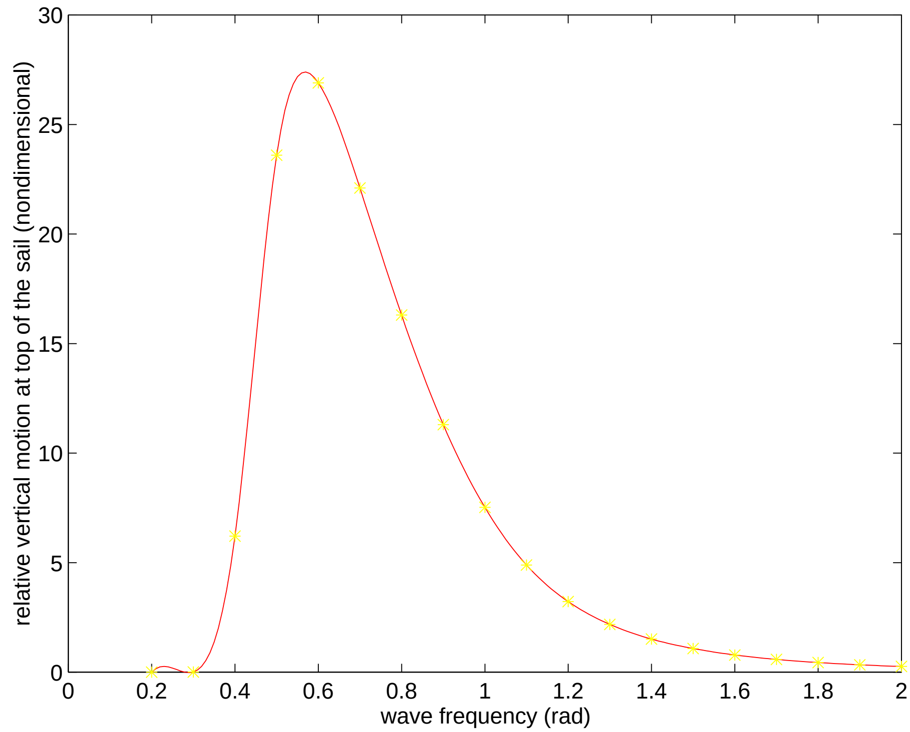

Figure 5. Spectrum of response for relative vertical motion

The average period between zero upcrossings was determined by equation (22), and the number between zero upcrossings per unit time is

$$N_z^R = \frac{1}{2\pi}\sqrt{\frac{m_2^R}{m_0^R}} \ , \tag{27}$$

where m_0^R, m_2^R are the moments of the particular response *R*, whose spectral density is given by equation (26). Equation (27) can be generalized for the case of the average number of upcrossings above a specified level α as in

$$N_{z,\alpha}^R = \frac{1}{2\pi}\sqrt{\frac{m_2^R}{m_0^R}}\exp\left(-\frac{\alpha^2}{2m_0^R}\right). \tag{28}$$

Equation (28) can be utilized to determine such events deck wetness and bow slamming for a surface ship or periscope submergence and sail broaching for a near surface submarine. If f represents height of the periscope over calm sea surface level, the number of periscope submergence events per hour is

$$N_p = 3600 \frac{1}{2\pi} \sqrt{\frac{m_2}{m_0}} \exp\left(-\frac{f^2}{2m_0} \right) , \qquad \textbf{(29)}$$

where m_0 , m_2 are the moments of the vertical relative motion spectrum at periscope. The same equation can be used to estimate the frequency of sail broaching, with f substituted by the distance between top of the sail and encountered wave surface. Of course, m_0 , m_2 are now the moments of the relative motion spectrum at top of the sail. When we calculate the frequency of vehicle collision with the sea-bed for shallow water operations this time m_0 , m_2 are the moments of the absolute motion spectrum at either the bow or the stern of the submarine and f is its clearance from the sea-bed.